

# Local unitary equivalence and distinguishability of arbitrary multipartite pure states

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We give an universal algorithm for testing the local unitary equivalence of states for multipartite system with arbitrary dimensions.

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Entanglement in multipartite quantum systems, due to its importance in quantum information theory, became a topic of numerous investigations (see [1] for a recent comprehensive review of various aspects of characterization and applications of multipartite entangled states). Until now, however, the efforts did not bring final solutions to many fundamental problems. One of the question is the classification of states which are interconvertible by local unitary transformations, i.e. operations on the whole systems composed from unitary actions (purely quantum evolutions) each of which is restricted to a single subsystem. To appreciate the importance of such an experimental setting let us remind that it is a basis for such spectacular applications of quantum information technologies like teleportation or dense coding where the fundamental parts of experiments consist of manipulations restricted to parts of the whole systems in distant laboratories.

Recently, the problem of checking local unitary (LU) equivalence of qubit systems was solved in [2, 3]. In this letter we present a geometric approach to the problem which supersedes previous ones by its applicability to systems of arbitrary dimensions treating on equal footing qubits and qudits. The same geometric formulations allows also to answer in arbitrary dimensions the question of distinguishing multipartite states by local measurements (i.e. measurements constrained to single subsystems). By taking advantage of relations between both concepts we are able to give an universal algorithm for checking LU-equivalence of pure states in arbitrary dimensions. Despite a rather abstract geometrical picture underlying our approach the final algorithms for checking equivalence and distinguishability are easy in applications.

Consider a quantum system consisting of  $M$  identical subsystems with the Hilbert space  $\mathcal{H} = \otimes_{i=1}^M \mathcal{H}_i$  where each  $\mathcal{H}_i$  is  $N$ -dimensional dimensions of the subsystems can be done along the same lines, albeit with . We say that two pure states  $|v_1\rangle \in \mathcal{H}$  and  $|v_2\rangle \in \mathcal{H}$  are locally unitary equivalent (LU-equivalent) if and only if there exist unitary operators  $U_i \in U(N)$  such that

$$|v_1\rangle = U_1 \otimes \dots \otimes U_M |v_2\rangle. \quad (1)$$

We call states  $|v_1\rangle$  and  $|v_2\rangle$  indistinguishable by local

measurements if and only if

$$\langle v_1 | A v_1 \rangle = \langle v_2 | A v_2 \rangle \quad \forall A, \quad (2)$$

where  $A$  is a local hermitian operator, i.e. it is real combinations of  $A_1 \otimes I \otimes \dots \otimes I, \dots, I \otimes \dots \otimes I \otimes A_M$  and  $A_i$  are  $N \times N$  hermitian matrices. Each operator correspond to a measurement performed on a single subsystem. Physically, we may imagine that subsystems are located in different laboratories, each local measurement is performed on a separate copy of the state in question and by pooling results obtained from such measurement we want to distinguish different states.

Pure states are, in fact, points in the projective space  $\mathbb{P}(\mathcal{H})$  rather than vectors in  $\mathcal{H}$  – two vectors  $|v\rangle, |w\rangle \in \mathcal{H}$  represent the same state if and only if  $|v\rangle = z|w\rangle$ , where  $z \in \mathbb{C}$ . Identification of all vectors differing by a multiplicative complex constant gives a point in  $\mathbb{P}(\mathcal{H})$ . We will denote by  $\pi(|v\rangle)$  the point in the projective space corresponding to  $|v\rangle \in \mathcal{H}$ , i.e.  $\pi$  denotes the canonical projection from  $\mathcal{H}$  to  $\mathbb{P}(\mathcal{H})$ . The action of the unitary group on  $\mathcal{H}$  as in (1) translates *via*  $\pi$  to an action on  $\mathbb{P}(\mathcal{H})$ . We can thus reformulate the definition of LU-equivalent states in the language of the projective space. Two pure states  $|v_1\rangle$  and  $|v_2\rangle$  are locally unitary equivalent (LU-equivalent) if and only if there exist special unitary operators  $U_i \in \text{SU}(N)$  such that

$$\pi(|v_1\rangle) = \pi(U_1 \otimes \dots \otimes U_M |v_2\rangle). \quad (3)$$

In more technical terms LU-equivalent states lie on the same orbit of  $G = \text{SU}(N) \otimes \dots \otimes \text{SU}(N)$  action on  $\mathbb{P}(\mathcal{H})$  and checking the LU-equivalence can be reduced to investigations of the structure of such an orbits. We restricted the group to  $\text{SU}(N)$  since vectors differing by a phase factor are indistinguishable on the projective space level. Furthermore, two states represented by arbitrary (not necessary norm one) vectors  $|v_1\rangle$  and  $|v_2\rangle$  are indistinguishable by local measurements if and only if they fulfill (2) with  $|v_i\rangle$  substituted by  $\frac{|v_i\rangle}{\sqrt{\langle v_i | v_i \rangle}}$ .

The study of orbits of the group  $G$  in the projective space  $\mathbb{P}(\mathcal{H})$  is facilitated by the existence of some additional structures on  $\mathbb{P}(\mathcal{H})$  and  $G$  which we now briefly describe.

The projective space  $\mathbb{P}(\mathcal{H})$  is a symplectic manifold. It means that there exists a differential two-form  $\omega$  on

$\mathbb{P}(\mathcal{H})$  which is closed ( $d\omega = 0$ ) and nondegenerate. To define  $\omega$  we have to determine its action on a pair of arbitrary vectors  $A_x, B_x$  tangent to  $\mathbb{P}(\mathcal{H})$  in an arbitrary point  $x$ . Now, each tangent vector  $A_x$  is a vector tangent to the curve  $t \mapsto \pi(\exp(tA)|v\rangle)$  at  $t = 0$ , where  $A$  is some element of the Lie algebra of the full unitary group  $U(\mathcal{H})$  [in our case by a choice of a basis in  $\mathcal{H}_i$  it can be identified with  $U(N^M)$ ], and  $|v\rangle$  is a vector in  $\mathcal{H}$  such that  $\pi(|v\rangle) = x$ . We have then

$$\omega(A_x, B_x) = -\text{Im} \frac{\langle Av|Bv\rangle}{\langle v|v\rangle} = \frac{i}{2} \frac{\langle [A, B]v|v\rangle}{\langle v|v\rangle}. \quad (4)$$

It is easy to check that the above definition is correct, i.e. it does not depend on the choice of  $|v\rangle$  and moreover  $\omega$  is invariant with respect to the action of  $U(\mathcal{H})$  on  $\mathbb{P}(\mathcal{H})$ , i.e. the action of  $U(\mathcal{H})$  is *symplectic*. This fact is a basis for another canonical construction in symplectic geometry, namely the *moment map* [4]. Let us denote by  $\mathfrak{u}(\mathcal{H})$  the Lie algebra of the unitary group  $U(\mathcal{H})$ , i.e. the algebra of all anti-Hermitian operators acting on  $\mathcal{H}$  and by  $\mathfrak{u}^*(\mathcal{H})$  the space dual to  $\mathfrak{u}(\mathcal{H})$ , i.e. the space of all linear operators on  $\mathfrak{u}(\mathcal{H})$ . The latter can be identified with the space of all Hermitian operators (observables) on  $\mathcal{H}$  and the action of  $Y \in \mathfrak{u}^*(\mathcal{H})$  on  $X \in \mathfrak{u}(\mathcal{H})$  is given by  $\langle Y, X \rangle = \text{Tr}(XY)$ , where in order to calculate the trace we may choose an arbitrary basis in  $\mathcal{H}$  and treat all operators as appropriate matrices.

Instead of giving the formal definition of the moment map  $\mu$  in the most general setting (see [4]) we write explicitly an expression valid in the considered case where  $\mu : \mathbb{P}(\mathcal{H}) \rightarrow \mathfrak{u}^*(\mathcal{H})$  can be shown to read as

$$\langle \mu(x), A \rangle = \frac{i}{2} \frac{\langle v|Av\rangle}{\langle v|v\rangle}, \quad x = \pi(|v\rangle). \quad (5)$$

Besides of its fundamental mathematical origin the idea of moment map has a natural physical interpretation. Namely to any state  $|v\rangle$  we assign a linear functional  $\mu(\pi(|v\rangle)) \in \mathfrak{u}^*(\mathcal{H})$  which encodes information about all expectation values of all observables in the state  $|v\rangle$ . In the following we will be interested in the restrictions of  $\omega$  and  $\mu$  to an orbit  $G.x$  of the group  $G$  through a point  $x \in \mathbb{P}(\mathcal{H})$ . They can be calculated as in (4) and (5) but now  $A$  and  $B$  are restricted to elements of  $\mathfrak{g} = \mathfrak{su}(N) \oplus \dots \oplus \mathfrak{su}(N)$ , i.e., the Lie algebra of  $G$ . In case of the moment map it means that to every state  $|v\rangle$  we assign an element of  $\mathfrak{g}^*$  which encodes expectation values of Hermitian operators of the following type

$$Y_1 \otimes I \otimes \dots \otimes I + \dots + I \otimes \dots \otimes I \otimes Y_M. \quad (6)$$

It means we extract from state  $|v\rangle$  the information about local measurements as we only control operators of (6) type.

Observe that we can identify two natural actions of the group  $U(\mathcal{H})$ . The first one is the action on  $\mathbb{P}(\mathcal{H})$  induced from the ordinary action on  $\mathcal{H}$  itself, whereas

the second is the *coadjoint action* on  $\mathfrak{u}^*(\mathcal{H})$  defined as  $Y \mapsto UYU^\dagger$ . Both are intertwined by the moment map. It can be directly checked using (5) that  $\mu$  is *equivariant*,

$$\langle \mu(\pi(U|v\rangle)), A \rangle = \langle U\mu(\pi(|v\rangle))U^\dagger, A \rangle. \quad (7)$$

The moment map projects thus the orbit through  $x = \pi(|v\rangle)$  on the orbit of the coadjoint action through  $\mu(\pi(|v\rangle))$ . As already mentioned we may restrict the reasoning to the actions of  $G$  which is a subgroup of  $U(\mathcal{H})$ . Generally the mapping by the moment map will not be a diffeomorphism between the orbit  $G.x$  and the coadjoint orbit through  $\mu(\pi(|v\rangle))$ . The latter operator can be stabilized by some subgroup  $\text{Stab}(\mu(\pi(|v\rangle)))$  of  $G$ , i.e.,  $U\mu(\pi(|v\rangle))U^\dagger = \mu(\pi(|v\rangle))$  for some nontrivial  $U$ . Likewise, we may have  $\pi(V|v\rangle) = \pi(|v\rangle)$  for some nontrivial unitaries  $V$  forming a subgroup  $\text{Stab}(\pi(|v\rangle))$  of  $G$ . In general the stabilizers  $\text{Stab}(\pi(|v\rangle))$  and  $\text{Stab}(\mu(\pi(|v\rangle)))$  are not equal, so the mapping is not diffeomorphic. The other facet of this fact is that the form  $\omega$  restricted to  $G.x$  is no longer nondegenerate;  $G.x$  is not a symplectic space. The difference between dimensions of both stabilizer subgroups conveys an important information useful in deciding the LU-equivalence as will be shown in the second example further in the text.

The fact that  $\mu$  maps orbits of  $G$  in  $\mathcal{H}$  on orbits of the coadjoint action is crucial for identifying states on the same orbit. The relevant observation is that each coadjoint orbit intersects the subspace in  $\mathfrak{g}^*$  which is dual to the maximal commutative subalgebra of  $\mathfrak{g}$  [4]. In our case this subspace consists of operators of the form (6) with  $Y_i$  diagonal. Let us explain how this fact is reflected in the present setting. To this end we consider a state

$$|v\rangle = \sum_{i_1, \dots, i_M=1}^N C_{i_1 \dots i_M} |i_1\rangle \otimes \dots \otimes |i_M\rangle. \quad (8)$$

From the coefficients  $C_{i_1 \dots i_M}$  we can build  $M$  following  $N \times N$  Hermitian positive semidefinite matrices

$$(C_k)_{\hat{i}_k \hat{j}_k} = \bar{C}_{i_1 \dots \hat{i}_k \dots i_M} C_{i_1 \dots \hat{j}_k \dots i_M}, \quad (9)$$

where by the overbar we denoted the complex conjugation and the summation over repeating indices is assumed. The orbit of  $G$  through  $x = \pi(|v\rangle)$  is now mapped by  $\mu$  on some coadjoint orbit and the fact that the latter contains a point (6) with  $Y_i$  diagonal means, after translating back to the orbit  $G.x$ , that it contains a point  $x' = \pi(|v'\rangle)$ ,  $|v'\rangle = \sum_{i_1, \dots, i_M=1}^N C'_{i_1 \dots i_M} |i_1\rangle \otimes \dots \otimes |i_M\rangle$  for which the corresponding matrices  $C'_k$  are diagonal,  $C'_k = \text{diag}(p_{1k}^2, \dots, p_{Nk}^2)$ . The numbers  $p_{lk}^2$  have a straightforward interpretation as probabilities to find the  $k$ -th subsystem in the state  $|l\rangle$  while neglecting other  $M - 1$  subsystems (see [5] for detailed calculations). Let us remark that in the case of two subsystems,  $M = 2$ , the whole procedure leads to the familiar Schmidt decomposition which allows for a straightforward identification of

LU-equivalent states for bipartite systems. There is no useful generalization of the Schmidt decomposition for more than two subsystems, nevertheless, the following reasoning shows how to overcome this obstacle.

To pass from  $|v\rangle$  to  $|v'\rangle$  one has to act on  $|v\rangle$  by  $\tilde{U}_1^T \otimes \dots \otimes \tilde{U}_M^T$  ( $T$  means the transposition), where  $\tilde{U}_k \in \text{SU}(N)$  are such that  $\tilde{U}_k^\dagger C_k \tilde{U}_k = \text{diag}(p_{1k}^2, \dots, p_{Nk}^2)$ . It is always possible to find such unitaries since, as already mentioned,  $C_k$  are Hermitian. Moreover one can always choose  $\tilde{U}_k$  in a such a way that  $p_{1k}^2 \geq \dots \geq p_{Nk}^2$  and in the following we assume it was done. In fact all intersections of the coadjoint orbit with (6) with diagonal  $Y_i$  differ only by ordering of the diagonal elements. We avoid thus this ambiguity by fixing the order.

The image of  $\pi(|v'\rangle)$  under the action of the moment map  $\mu$  can be calculated [5]. It has the form (6) with

$$Y_k = \alpha \text{diag}\left(-\frac{1}{N} + p_{1k}^2, \dots, -\frac{1}{N} + p_{Nk}^2\right), \quad (10)$$

where  $\alpha$  is some appropriate const. Going back to the above mentioned physical interpretation of  $\mu$  we see that in fact moment map encodes information about the state  $|v'\rangle$  contained in local measurements as it is determined only by the local probabilities  $p_{kl}^2$ .

The matrices  $C'_k$  and, consequently  $Y_k$  have, in general, degenerate spectra, i.e, several  $p_{lk}$  can repeat in (10). Let us denote by  $\nu_k$  the number of different eigenvalues of  $C'_k$  and by  $m_{k,n}$  the multiplicity of the  $n$ -th eigenvalue. It is now easy to show [5] that the stabilizer  $\text{Stab}(\mu(\pi(|v'\rangle)))$  consists of  $U_1 \otimes \dots \otimes U_M \in G$  where

$$U_k = \begin{pmatrix} u_{k,0} & & & \\ & u_{k,1} & & \\ & & \ddots & \\ & & & u_{k,\nu_k} \end{pmatrix}, \quad (11)$$

with  $u_{k,n} \in \text{U}(m_{k,n})$ . The equivariance (7) of  $\mu$  implies that  $\text{Stab}(\pi(|v'\rangle)) \subset \text{Stab}(\mu(\pi(|v'\rangle)))$ . Consequently, for any  $|v\rangle$  the corresponding state  $|v'\rangle$  is given up to the action of  $\text{Stab}(\mu(\pi(|v'\rangle)))$ , which is explicitly known.

We thus arrive at the following algorithm to check if two states  $|v_1\rangle$  and  $|v_2\rangle$  are LU-equivalent.

- For both states compute the matrices  $C_k$  (9) and check whether they have pairwise the same ordered spectra. If this is not the case the states are not LU-equivalent.
- If the first condition is fulfilled use the unitary matrices  $\tilde{U}_k$  diagonalizing  $C_k$  for each state to find  $|v'_1\rangle$  and  $|v'_2\rangle$ . If they are equal then  $|v_1\rangle$  and  $|v_2\rangle$  are LU-equivalent, otherwise they are LU-equivalent if and only if there exist  $U_1 \otimes \dots \otimes U_M \in \text{Stab}(\mu(\pi(|v'_1\rangle)))$  such that  $|v_1\rangle = U_1 \otimes \dots \otimes U_M |v_2\rangle$ .

In case of generic states  $\text{Stab}(\mu(\pi(|v'_i\rangle)))$  consists of diagonal matrices  $\text{diag}(e^{i\phi_1}, \dots, e^{i\phi_N})$  and above method

can be easily applied. For other states our method also simplifies computations since we no longer need to use the full  $U(N) \otimes \dots \otimes U(N)$  group but can restrict our attention to an appropriate subgroup. The worst case is when the state  $|v\rangle$  is such that all  $C_k$  are  $\frac{1}{\sqrt{N}}\mathbb{I}$ . Then the action of  $U(N) \otimes \dots \otimes U(N)$  on such state gives a state for which again all  $C_k$  are normalized identity matrices.

Our approach gives also a beautiful mathematical characterization of states which are indistinguishable by local measurements. As we noticed before for any state  $|v\rangle$  the moment map  $\mu$  assigns an element of  $\mathfrak{g}^*$  which encodes all expectation values of local measurements in state  $|v\rangle$ . In other words all states that have the same expectation values for local measurements are sent by the moment map to the same point of  $\mathfrak{g}^*$  – they constitute a fiber over this point. To examine if states  $|v'_1\rangle$  and  $|v'_2\rangle$  obtained in the second step of the above algorithm are indistinguishable by local measurements it is enough to check if ordered spectra of corresponding matrices  $C_k$  are the same. An intimate connection between LU-equivalence and indistinguishability by local measurements appears to be helpful in deciding the former in various situations, also in the ‘worst case’ mentioned above, as we show in the second example below.

A detailed analysis showing how to apply our method to different kind of states will be published elsewhere [6]. Here we give only two simple examples. The first one shows how the proposed algorithm works in a generic case and involves three qutrits. The Hilbert space is  $\mathcal{H} = \mathbb{C}^3 \otimes \mathbb{C}^3 \otimes \mathbb{C}^3$  and has the real dimension  $\dim_{\mathbb{R}}(\mathcal{H}) = 54$ . The dimension of the projective space  $\mathbb{P}(\mathcal{H})$  is thus  $\dim_{\mathbb{R}}(\mathbb{P}(\mathcal{H})) = 52$ . We are interested in the orbits of  $G = \text{SU}(3) \otimes \text{SU}(3) \otimes \text{SU}(3)$  - action on  $\mathbb{P}(\mathcal{H})$ . The Lie algebra  $\mathfrak{g}$  of  $G$  is spanned by  $A \otimes I \otimes I$ ,  $I \otimes A \otimes I$ , and  $I \otimes I \otimes A$  with  $A \in \mathfrak{su}(3)$  and  $\dim_{\mathbb{R}}(\mathfrak{g}) = 24$ . Let us consider two states

$$|\Psi\rangle = \alpha|W\rangle + \beta|222\rangle = \alpha(|001\rangle + |010\rangle + |100\rangle) + \beta|222\rangle, \\ |\Phi\rangle = \sqrt{2}\alpha|000\rangle + \alpha|111\rangle + \beta|222\rangle,$$

where  $3|\alpha|^2 + |\beta|^2 = 1$  and  $|\beta| < |\alpha|$ . Direct computations show that for both  $|\Psi\rangle$  and  $|\Phi\rangle$  we get:

$$C_1 = C_2 = C_3 = \begin{pmatrix} 2|\alpha|^2 & 0 & 0 \\ 0 & |\alpha|^2 & 0 \\ 0 & 0 & |\beta|^2 \end{pmatrix}. \quad (12)$$

Since the corresponding matrices for both states are the same,  $|\Phi\rangle$  and  $|\Psi\rangle$  are indistinguishable by local measurements. Remember that if for at least one pair of the matrices  $C_i$  calculated for  $|\Phi\rangle$  and  $|\Psi\rangle$  the results differ the algorithm ends deciding that the states are not LU-equivalent. Since this is not the case we have to invoke the second step. The matrices (12) are already diagonal so we do not need to change the form of  $|\Psi\rangle$  and  $|\Phi\rangle$ . Let us thus look at the states that are LU - equivalent with

$|\Phi\rangle$  and for which matrices  $C_1, C_2, C_3$  are given by (12). Such states are generated by action of  $\text{Stab}(\mu(|\Phi\rangle))$  on  $|\Phi\rangle$ . But  $\text{Stab}(\mu(|\Phi\rangle))$  consists of matrices  $U_1 \otimes U_2 \otimes U_3$  where:

$$U_k = \begin{pmatrix} e^{i\phi_{k1}} & 0 & 0 \\ 0 & e^{i\phi_{k2}} & 0 \\ 0 & 0 & e^{-i(\phi_{k1}+\phi_{k2})} \end{pmatrix}, \quad (13)$$

The action of (13) on  $|\Phi\rangle$  gives

$$|\Phi'\rangle = \sqrt{2}\alpha e^{i\phi_1}|000\rangle + e^{i\phi_2}\alpha|111\rangle + e^{i\phi_3}\beta|222\rangle \quad (14)$$

where  $\phi_i$  are appropriate combinations of  $\phi_{ki}$ . It is easy to notice that  $|\Psi\rangle$  is not of the (14) form and in the effect states  $|\Psi\rangle$  and  $|\Phi\rangle$  are not  $LU$ -equivalent.

Fibers of the moment map consisting, as mentioned above, of locally indistinguishable states are given as common level sets  $\{f_A(x) = c_A\}$  of functions  $f_A(\pi(v)) = \langle \mu(|v\rangle), A \rangle$  [see (5)], where  $A \in \mathfrak{g}$  and  $c_A$  are some constants. An interesting question which one should ask is what is the relationship between the tangent space to the fiber of the moment map at given point  $x = \pi(|v\rangle)$  and the tangent space to the orbit of  $G$ -action at the same point. These two tangent spaces generate infinitesimal movements in  $\mathbb{P}(\mathcal{H})$  in the direction of states which are indistinguishable by local measurements from state  $x$  and, respectively,  $LU$ -equivalent with  $x$ . The tangent space to the fiber at a given point  $x = \pi(|v\rangle)$  is of course contained in the intersection of kernels of all 1-forms  $d_x f_A$ , where  $A$  is an element of  $\mathfrak{g}$ . It is very important to emphasize that it might be just a subspace of this intersection [6].

To continue our discussion we will need the following definition.

**Definition 1** *Let  $(V, \omega)$  be symplectic vector space and  $W \subset V$  its subspace. Then  $\omega$ -orthogonal complement of  $W$  is given by*

$$W^{\perp\omega} = \{v \in V : \omega(w, v) = 0, \forall w \in W\}. \quad (15)$$

In [6] we prove that the tangent space to the orbit of  $G$ -action at  $x$  is spanned by vectors which are  $\omega$ -orthogonal to the intersection of kernels of all 1-forms  $d_x f_A$  for  $A \in \mathfrak{g}$ . Knowing just these kernels we can thus find the dimension of an orbit of the  $G$ -action. The other facet of the above reasoning is that when the tangent space to a fiber of the moment map is contained in the tangent space to the orbit  $G.x$  then the local unitary equivalence is the same as the indistinguishability by local measurements, i.e., in this case we can restrict our method just to the comparison of ordered spectra of  $C_k$  matrices. In [5] we

prove that the dimension of the part of the moment map which is contained in the orbit of the  $G$ -action is equal to difference between dimension of stabilizers of  $\pi(|v\rangle)$  and  $\mu(\pi(|v\rangle))$  mentioned above.

The next example shows a practical application of the above observation. Let us consider a three-qubit state

$$|GHZ\rangle = \frac{1}{\sqrt{2}}(|000\rangle + |111\rangle) \quad (16)$$

Here  $\dim_{\mathbb{R}}(\mathbb{P}(\mathcal{H})) = 14$ . One calculates  $C_1 = C_2 = C_3 = \frac{1}{2}\mathbf{1}_{2 \times 2}$ . Hence  $\text{Stab}(\mu(|GHZ\rangle))$  is the whole local unitary group, i.e.,  $\text{Stab}(\mu(|GHZ\rangle)) = SU(2) \otimes SU(2) \otimes SU(2)$ , and its action on  $|GHZ\rangle$  is the whole orbit  $\mathcal{O}_{|GHZ\rangle}$ . Thus the part of the fiber of the moment map contained in the orbit  $\mathcal{O}_{|GHZ\rangle}$  is exactly the orbit  $\mathcal{O}_{|GHZ\rangle}$ . The intersection,  $K$ , of kernels of  $df_{A \otimes I \otimes I}$ ,  $df_{I \otimes A \otimes I}$ ,  $df_{I \otimes I \otimes A}$ , where  $A \in \mathfrak{su}(3)$  is seven dimensional. The dimension of the  $\omega$ -orthogonal complement of  $K$  is the dimension of the  $G$ -orbit through  $|GHZ\rangle$  and is equal  $\dim(\mathcal{O}_{|GHZ\rangle}) = 7$ . The maximal possible dimension of the fiber is also 7 since it is the dimension of  $K$ . This means that the whole fiber of the moment map is contained in  $\mathcal{O}_{|GHZ\rangle}$ . We infer thus that two three-qubit states for which  $C_1 = C_2 = C_3 = \frac{1}{2}\mathbf{1}_{2 \times 2}$  are automatically  $LU$ -equivalent, although the situation corresponds the ‘worst case’ when deciding the  $LU$ -equivalence should be the most difficult.

The presented examples do not exhaust the variety of possible situations met when checking  $LU$ -equivalence [6]. They correspond to, in a sense, two extremal cases, and show clearly that dimensionality of subsystems does not play a crucial role in applications, although, admittedly can make calculations more cumbersome.

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